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One-dimensional flow in an ionizing gas

By D. S. BUTLER†
Department of Mathematics, University of Strathclyde, Glasgow

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In order to gain insight into the macroscopic behaviour of gas interacting with electric and magnetic fields, while its degree of ionization is changing, the one-dimensional unsteady flow of gas with temperature-dependent conductivity is investigated. A simple continuum model is used for the gas. It is assumed to be compressible and to obey polytropic gas laws and its conductivity is supposed to change discontinuously at a critical temperature. Particular attention is devoted to the various types of transition boundaries which can occur as the conductivity changes, and it is shown that complete flow patterns can be constructed with these as elements. A wide variety of transitions are possible and some of these have remarkable properties. For example, it is shown that discontinuous de-ionising fronts can exist and are expansive in nature. This contrasts sharply with the situation in conventional gas dynamics and magneto-gasdynamics where only compressive discontinuities can exist.

1. Introduction

The purpose of this paper, which first appeared as a R.A.R.D.E. report (Butler 1963), is to study the interaction between a magnetic field and a gas in which ionization or de-ionization processes may be taking place. For simplicity, the discussion is restricted to purely one-dimensional unsteady flows with transverse magnetic fields, and, to avoid the complexity of the ionization process in a real gas (see, for example, Goldsworthy 1958), an idealized model is used.

The model is that used by Kulikovsky & Lyubimov (1959) for their work on the structure of ionizing shock waves. It is assumed that, at temperatures below some critical value T^*, the gas is non-conducting but at this temperature it becomes highly conducting, so that for $T > T^*$ it can be regarded as a perfect conductor. It is also assumed that there is no change in the internal energy of the gas as it ionizes, so that the same polytropic gas laws can be used throughout.

A discontinuous conductivity-temperature relation does not appear to be too serious an approximation to the behaviour of a real gas, since the conductivity of most gases rises very rapidly once a significant amount of ionization has occurred. Furthermore, it is reasonable to expect that using a discontinuous relation, in place of a steep but continuous one, will not seriously alter the overall behaviour of the gas, but will merely concentrate conductivity changes, which

† Formerly at the Royal Armament Research and Development Establishment, Fort Halstead, Sevenoaks, Kent.

might otherwise be spread over small bands. The neglect of ionization energy is open to more serious objection, since this involves a change in the overall conservation laws on which the flow equations are based. However, the effect of ionization energy and the increase in particle density can probably be simulated by allowing the adiabatic index γ to vary. This is unlikely to affect the arguments seriously.

The equations governing the one-dimensional motion of a gas interacting with magnetic and electric fields are set out in §2. It is assumed that the particle velocity, magnetic field and electric field vectors are mutually perpendicular. There are three fluid-dynamic equations, which impose the conditions of conservation of mass, momentum and energy on the fluid. In addition there are Maxwell's equations and Ohm's law which must be satisfied by the magnetic and electric fields and the current density. The non-relativistic forms of the fluid-dynamic equations are used, but, when the gas is not fully conducting, the displacement-current term in Maxwell's equations is retained. This procedure is not completely consistent since the non-relativistic equations are only valid for $v \ll c_0$, where v is a typical fluid velocity and c_0 is the speed of light, and under these conditions displacement currents may be neglected in general. However, it is found that electromagnetic waves have a crucial effect in some cases and they must therefore be included. It should always be borne in mind that the equations derived are only valid in the limit as $c_0 \to \infty$.

In the non-conducting régime the flow and the magnetic field are uncoupled. The motion of the gas is governed by the conventional equations of unsteady flow and the Maxwell equations govern the electric and magnetic fields. The governing equations in the fully-conducting régime are the magneto-gasdynamic equations for a perfectly conducting gas and are fully coupled.

Although the conductivity-temperature relation is assumed to be discontinuous, all transitions between the fully-conducting and non-conducting régimes are not necessarily instantaneous. In reality the relation between conductivity and temperature is not discontinuous but merely steep. In these circumstances a small change in temperature will produce a large change in conductivity, so that during a transition the temperature will remain approximately constant, while the conductivity changes.

In order to make the discontinuous model self consistent, an additional flow régime must be introduced, which will approximately reproduce this behaviour during slow transitions between the non-conducting and the fully-conducting régimes or vice versa. In this régime the temperature is constant and equal to the critical value, while the conductivity takes any positive value which is dictated by the governing equations. This leads to a consistent system of equations since the freedom that is lost by imposing a fixed temperature is recovered by allowing the conductivity to be variable.

The equations relevant to the non-conducting and fully-conducting régimes are well known. Both sets are hyperbolic. The isothermal régime is also hyperbolic in spite of finite conductivity. This is not so surprising when it is remembered that the full magneto-gasdynamic equations with finite conductivity depending on the thermodynamic variables are also hyperbolic if the speed of light is finite.

The parabolic diffusion terms only arise as an approximation to electromagnetic wave propagation in the limit as $c_0 \to \infty$. However, in the present case the two slower characteristics remain even when $c_0 \to \infty$.

Before the solution of a particular boundary-value problem can be attempted, the appropriate flow configuration must be determined. This is done by piecing together regions of different types of flow and matching them across suitable free boundaries. It is clearly important to know what types of boundaries are possible and what conditions apply across them. §§3 and 4 of this paper are devoted to studying these boundaries.

Free boundaries can be divided into two classes, those which allow discontinuities in the flow variables themselves and those which allow only discontinuities in derivatives. Boundaries belonging to these classes will be described as discontinuous and continuous respectively. If a particular type of boundary is to exist, it must satisfy two requirements. First, it must allow consistent solutions of the flow equations in its neighbourhood. This will usually imply that the velocity at which it moves is, within certain limits, determined by the velocities corresponding to the characteristic curves in its neighbourhood. For continuous boundaries the second requirement is that the governing equations should allow a discontinuity in derivatives. Discontinuous boundaries must have a structure which satisfies more detailed physical equations which permit dissipation.

It is well known that, in both the non-conducting and the fully-conducting régimes, continuous boundaries can occur at characteristics. However, in the case of a mixed-conducting and non-conducting flow these boundaries can also occur when $T = T^*$. In these cases the governing equations change discontinuously and there will normally be a corresponding change in derivatives. The second requirement is therefore satisfied. The implications of the first requirement are considered in §3. Electromagnetic-wave propagation plays a vital role in this analysis.

Discontinuous boundaries can be of two types, those which move with the fluid and those which do not. Those of the first type are contact discontinuities. These can exist either with the same flow régime on both sides, or they can separate regions of different types of flow. Across contact discontinuities the flow velocity and the total pressure (thermodynamic plus magnetic) must be continuous. For contact surfaces with non-conducting flow on both sides the magnetic field must also be continuous. All possible types of contact discontinuities satisfy the consistency requirement.

Two well-known types of discontinuity moving through the fluid can occur. These are conventional fluid-dynamic shocks with non-conducting fluid on both sides and shocks with fully ionized flow on both sides (Marshall 1955). Discontinuities across which the gas changes from non-conducting to fully conducting or vice versa can also occur. The structure of these is studied in §4. This discussion is based, for concreteness, upon the Navier–Stokes equations for a viscous conducting fluid, and follows the same lines as that used by Kulikovsky & Lyubimov (1959).

The arguments do not seem to depend vitally on the particular dissipation mechanism introduced. Indeed, as Kulikovsky & Lyubimov show in the particular case they consider, essentially the same results are obtained if heat conduction is used instead of viscosity. In principle it is also possible to include heat conduction and viscosity simultaneously, although this would lead to practical difficulties since solution curves in a 3-space would have to be considered. It seems likely that the conclusions are valid under conditions of greater generality than those for which they are proved.

The study of the discontinuity structure shows that four distinct types of discontinuity are possible. The first is an ionization front or shock, with non-conducting gas moving with supersonic velocity relative to the front on the upstream side, and fully-conducting subsonic flow downstream. This shock is the one studied by Kulikovsky & Lyubimov (1959). The second type has non-conducting flow on both sides. Internally the temperature momentarily exceeds the critical value and currents flow, so that there is an overall increase in magnetic field. This type of front is effectively an arc moving in a magnetic field. The third and fourth types are de-ionizing fronts and are expansive rather than compressive in nature. In both cases the density, pressure and temperature of the gas decrease while the entropy increases. The third type is subsonic relative to both the flow ahead and behind, and the magnetic field strengthens as the gas passes through. The fourth type is supersonic on both sides and involves a decrease in magnetic field.

The jump conditions across conventional shocks are derived solely from the conservation laws for mass, momentum and energy. This is also true of shocks in highly conducting gases, except that the condition for conservation of transverse electric field is also required. Study of the structure of these shocks gives, at most, restrictive inequalities, which the states on either side must satisfy. For instance, Marshall (1955) showed that if there is no component of magnetic field in the direction of propagation the transverse magnetic field must always increase across a shock in fully-conducting gas. For fronts of types 1, 2 and 3 the study of the structure brings to light an additional relation between the states. This relation depends on the physical model used for the structure analysis. For the Navier–Stokes model it depends on the relative values of the diffusion coefficients for vorticity and magnetic field.

It is fortunate that these relations exist, since without them it would be impossible to obtain unique solutions to problems involving discontinuities. As it is, there are just the required number of jump conditions in each case and consistent solutions can be obtained. This feature seems to arise because there are different numbers of characteristics in the two regions to be connected. In the non-conducting régime there are five families of characteristics (the three characteristics of the flow equations plus the two characteristics of the electromagnetic-wave equations) and in the fully-conducting régime there are four (the particle paths are counted twice, since there are two relations along them).

The extra constraint coming from the structure properties arises because the magnetic field is not allowed to vary when the gas is non-conducting. This depends implicitly on the possibility of purely electromagnetic waves in non-conducting gas. Any non-uniformity in the magnetic field would result in electromagnetic-wave propagation tending to reduce the non-uniformity. If the velocity of light is effectively infinite, this adjustment takes place instantaneously. Electromagnetic waves, therefore, provide an additional mechanism for transition from one equilibrium state to another. There are altogether five mechanisms for readjustment; diffusion of mass, momentum, energy and magnetic field and electromagnetic waves. In the case of a conventional shock it does not matter how the available mechanisms are used (diffusion of mass, momentum and energy only in this case), since the same final state will be reached. In general, the gas will try to make the adjustment as quickly as possible. So it will start off by using principally the mechanism which allows adjustment to take place in the shortest distance (that is, the one with the smallest diffusion coefficient), the slower mechanisms will only be used when there is no alternative. For example, consider a conventional shock wave in a gas which allows diffusion of mass, momentum and energy, and suppose that the corresponding coefficients are D, ν and K, and satisfy $D,\ K \ll \nu$. Adjustments to the mass and energy flow can take place much more quickly than adjustments to the momentum. As a result, the mass and energy fluxes will virtually remain in equilibrium throughout. Momentum equilibrium can only be attained comparatively slowly, but since it must eventually be reached, the overall effect will be the same as it would if the coefficients had different relative values. In the case of an ionizing or de-ionizing shock, the situation is different because when the critical temperature is reached, the magnetic diffusion mechanism is replaced by the electromagnetic-wave mechanism or vice versa. There is also a change in the number of equilibrium conditions which the gas is trying to satisfy; the electrical equilibrium condition $\mathbf{E}+\mathbf{u}\times\mathbf{B} = 0$ is either gained or lost. The relative rates of adjustment alter the end states reached in this case. For a de-ionizing front, for example, once the gas becomes non-conducting, no further adjustment of the magnetic field is possible or indeed is needed, since electrical equilibrium is no longer necessary, and the amount of adjustment made before $T = T^*$ determines the final end state.

In the final section a few simple boundary-value problems are discussed. These illustrate how flow configurations can be constructed using the boundaries described in the earlier sections of this paper.

2. The equations

Cartesian axes are chosen so that the particle velocity is parallel to Ox, the electric field and current vectors to Oy and the magnetic field to Oz. The components of these vector quantities are then

$$\mathbf{u} = (u,0,0),\quad \mathbf{E} = (0,E,0),\quad \mathbf{J} = (0,J,0)\quad \text{and}\quad \mathbf{B} = (0,0,B).$$

The flow is assumed to be purely one dimensional with $\partial/\partial y = \partial/\partial z = 0$.

The equations governing the motion of the gas, assuming that it is polytropic, are

$$\rho_t + u\rho_x + \rho u_x = 0, \tag{2.1}$$

$$\rho(u_t + uu_x) + p_x = BJ, \tag{2.2}$$

$$p_t + up_x - a^2(\rho_t + u\rho_x) = (\gamma-1)\,J^2/\sigma, \tag{2.3}$$

where ρ is the density, a is the adiabatic sound speed and σ is the scalar electrical conductivity. Maxwell's equations and Ohm's law must also be satisfied. These give

$$B_t + E_x = 0, \tag{2.4}$$

$$\frac{1}{c_0^2} E_t + B_x = -\mu_0 J = -\mu_0 \sigma (E - uB), \tag{2.5}$$

where rationalized M.K.S. units are used and c_0 is the velocity of light and μ_0 the magnetic permeability. The term E_t/c_0^2 in equation (2.5) will be neglected eventually in order to be consistent with the non-relativistic form of equations (2.1), (2.2) and (2.3). It is retained temporarily since it throws light on the physical nature of the problem.

The conductivity σ is a given function of temperature T. It is assumed that the form of this function allows the temperature range to be divided into three régimes (figure 1). In régime 1, $T < T^*$ and $\sigma < \sigma_1$, where the magnetic Reynolds number based on $\sigma_1 (\mu \sigma_1 UL)$ is small compared to unity. The flow is then effectively non-conducting. In régime 3, $T > T^* + \Delta T$ and $\sigma > \sigma_2$ where $\mu \sigma_2 UL \gg 1$ and the flow is effectively 'infinitely conducting'. In régime 2, $T^* < T < T^* + \Delta T$ and $\sigma_1 < \sigma < \sigma_2$ but $\Delta T / T^* \ll 1$ so that the flow is effectively isothermal.

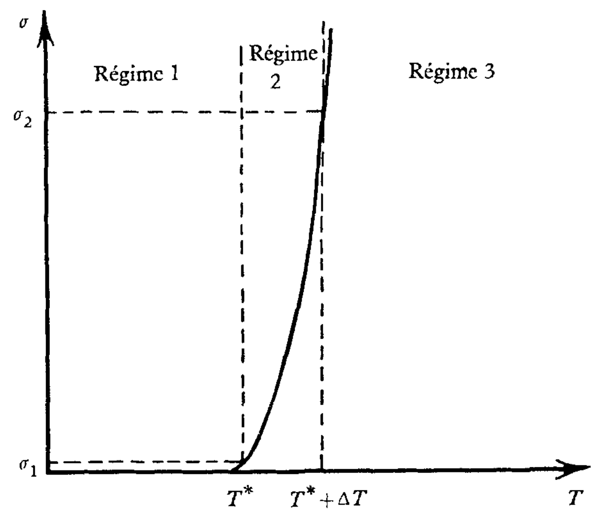

FIGURE 1. A typical conductivity-temperature relation.

In régime 1, $J = 0$ and the five differential equations (2.1) to (2.5) decouple into two sets, the unsteady-flow equations and the electromagnetic-wave equations. These equations involve five dependent variables ρ, u, p, B and E (a may be eliminated by using $a^2 = \gamma p / \rho$) and are hyperbolic with five families of characteristics. These characteristics are

$$\left.\begin{aligned} dx/dt &= u \pm a, \text{ on which } dp \pm \rho a\, du = 0, \\ dx/dt &= u, \text{ on which } dp - a^2 d\rho = 0, \\ dx/dt &= \pm c_0, \text{ on which } dB \pm \frac{1}{c_0} dE = 0. \end{aligned}\right\} \tag{2.6}$$

In régime 2, $T = T^*$ and therefore $p/\rho =$ a constant (say C^2). J and σ are eliminated from equations (2.2) and (2.3) by using (2.5) to give a set of four differential and one algebraic equations for the five dependent variables ρ, u, p, B and E. These are:

$$\left.\begin{aligned} &\rho_t + u\rho_x + \rho u_x = 0, \\ &\rho(u_t + uu_x) + p_x + \frac{B}{\mu_0}\left(B_x + \frac{1}{c_0^2}E_t\right) = 0, \\ &p_t + up_x - a^2(\rho_t + u\rho_x) + \frac{(\gamma-1)}{\mu_0}\left(B_x + \frac{1}{c_0^2}E_t\right)(E - uB) = 0, \\ &B_t + E_x = 0, \\ &p/\rho = C^2 = \frac{a^2}{\gamma}. \end{aligned}\right\} \quad (2.7)$$

Solutions must also satisfy the restriction

$$\left(B_x + \frac{1}{c_0^2}E_t\right)(E - uB) < 0, \quad (2.8)$$

since violation of this condition would imply negative conductivity. The equations (2.7) are hyperbolic and have four families of characteristics,

$$dx/dt = \pm c_0,$$

on which

$$c_0(c_0 \mp u)\rho\,du \pm c_0 C^2 d\rho$$
$$+\frac{1}{\mu_0}[\{\pm c_0 C^2 - u(c_0 \mp u)^2\}B - \{C^2 - (c_0 \mp u)^2\}E][dB \pm (1/c_0)\,dE] = 0, \quad (2.9)$$

and $dx/dt = u + \lambda$, on which $\lambda\rho\,du + dp = 0$, where λ is either root of

$$\lambda^2(E - uB)/C^2 + \lambda B - (E - uB) = 0.$$

Both roots are always real. It should be remembered that equation (2.9) is only meaningful in the limit as $c_0 \to \infty$.

In régime 3, σ is very large, and in the limit as $c_0 \to \infty$ the well-known equations for a perfectly-conducting gas are obtained. These consist of four partial differential equations and one algebraic equation ($E - uB = 0$) for the five dependent variables (ρ, u, p, B, E). There are three characteristics, on one of which (the particle path) there are two differential relations between the dependent variables:

$$dx/dt = u \pm c, \text{ on which } (B/\mu_0)\,dB + dp \pm \rho c\,du = 0 \text{ where } c^2 = a^2 + B^2/\rho\mu_0,$$

and $\quad dx/dt = u$, on which $dp - a^2 d\rho = 0$ and $(dB/B) - (d\rho/\rho) = 0$.

In each régime the total number of characteristics and algebraic relations is the same as the number of dependent variables. In principle, therefore, it is possible to solve complete initial-value problems for regions in which only one régime occurs.

3. Continuous transitions

Continuous transitions from one régime to another can occur across free boundaries at which $T = T^*$. At such boundaries it is assumed that the five dependent variables (ρ, u, p, B, E) are continuous but J and σ may be discontinuous. Considerable information about these boundaries can be obtained by adopting the following principle: flow patterns can only occur if they allow all the characteristic and algebraic conditions of the relevant régimes and complete initial data for ρ, u, p, B and E and the initial velocities of any free boundaries to be satisfied consistently. In order to test particular flow patterns it is assumed that a necessary condition for consistency is that it should be possible to devise a construction for the solution using all the relevant characteristic relations. The procedure used is illustrated by the following particular case. In essence it is similar to that used by Kontorovich (1958) to investigate the 'stability' of magneto-gasdynamic shocks, but the viewpoint is different.

Consider the configuration illustrated in figure 2. The heavily drawn curve represents the path of a transition from régime 1 to régime 3. Ahead of the transition (above the boundary in the figure) the flow satisfies the conditions for régime 1, while behind the boundary the flow is régime 3. Suppose that at $t = t_0, \rho, u, p, B, E$ and the velocity of the boundary at P' are given, and the solution at $t = t_0 + \Delta t$ is required. The solution at points whose domains of dependence are entirely within a single régime can be computed consistently since the number of characteristic and algebraic conditions is the same as the number of dependent variables. At the point P on the boundary there are six unknowns ρ, u, p, B, E and U, where U is the velocity of the boundary. These variables must satisfy two independent algebraic relations:

$$T = T^*,$$

and

$$E - uB = 0.$$

The characteristics through P are shown in figure 2 for the case

$$u + a < U < u + c.$$

The arrowheads indicate the direction of increasing t along the characteristics. CPG is the particle path, BP and DP are the characteristics $dx/dt = u \pm a$, AP and HP are $dx/dt = \pm c_0$, EP is the characteristic $dx/dt = u + c$ and PF is $dx/dt = u - c$ in régime 3. Five of these characteristics AP, BP, CP, DP and EP approach P as t increases, and on these curves the characteristic conditions can only be satisfied by adjusting the variables at P, since the variables are given at A, B, C, D and E. Seven conditions in all must therefore be satisfied and since there are only six unknowns this is impossible in general. Therefore a continuous transition from régime 1 to régime 3 moving with velocity U such that

$$u + a < U < u + c$$

does not in general give a consistent solution.

Consider now the case when $u < U < u+a$. The new configuration is shown in figure 3. Along the characteristic PD on which $dx/dt = u+a$, t increases as it leaves P. It is therefore no longer necessary to satisfy the condition on this

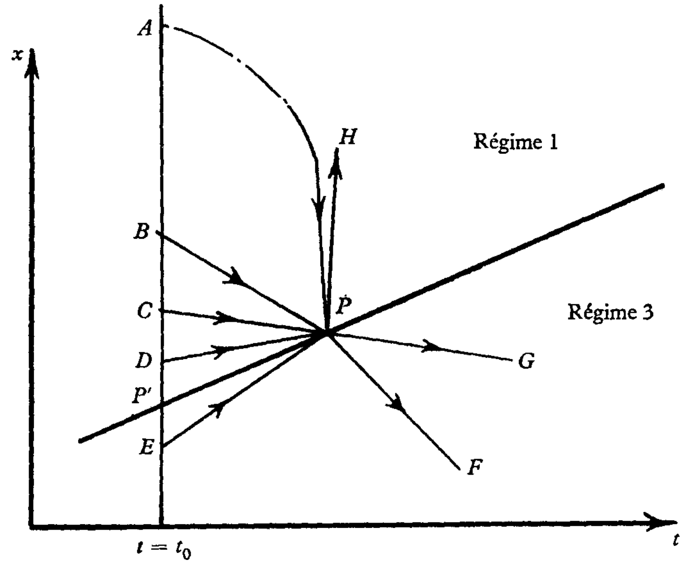

FIGURE 2. An example of a discontinuous transition between régimes, which leads to an inconsistent solution.

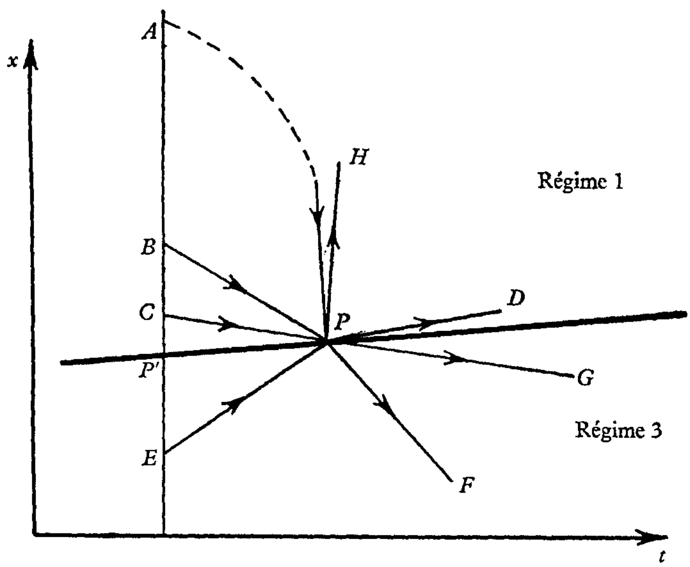

FIGURE 3. An example of a discontinuous transition, which leads to a consistent solution.

characteristic by adjusting variables at P. As a result the variables at P need only satisfy six conditions and, in general, a consistent solution can be obtained. If U is equal to one of the wave speeds then that characteristic condition must be counted. The cases $U = u$ or $U = u+a$ are therefore not possible.

By investigating all possible transitions in this way eight different types which lead to consistent solutions can be identified. Each type may face either forward or backward. The conditions under which each type can exist are shown in table 1.

Nature of transition	No. of types	Conditions on U (assuming $\lambda_2 > 0 > \lambda_1$)
	Ionizing	
$1 \rightarrow 2$	1	$u < U < u+a$, and $U < u+\lambda_2$
$2 \rightarrow 3$	1	$u+\lambda_2 < U < u+c$
$1 \rightarrow 3$	2	$u < U < u+a$, or $u+c < U < c_0$
	De-ionizing	
$2 \rightarrow 1$	1	$u+\lambda_2 < U < u+a$
$3 \rightarrow 2$	1	$u+\lambda_2 < U < u+c$
$3 \rightarrow 1$	2	$u+a < U < u+c$, or 'instantaneous' $U > c_0$

TABLE 1. Classification of continuous transitions.

4. Discontinuous transitions

The arguments of § 3 show that a continuous transition is only possible if the velocity of the transition front is within a certain range. Suppose that the early stages of a solution involve a continuous transition front which steadily changes its velocity until it reaches the end of this range. For example, the transition illustrated in figure 3 might be followed by a régime 3 compression wave which could accelerate it until $U = u+a$. If and when this stage is reached, it would no longer be possible to construct a solution with this continuous transition. One way in which the solution might be continued is by allowing the transition to divide into two with a régime 2 region interposed between the other two regions. Alternatively, the transition might subsequently take place across a discontinuity.

This situation is analogous to that which can occur in ordinary unsteady flow when a compression wave breaks to form a shock wave. In this case characteristics of the same family tend to cross, so that there may be more than one through a particular point. When this happens it is no longer possible to find a consistent solution without introducing a shock.

Discontinuous transitions may move with the gas so that they are contact surfaces. The conservation laws demand that, across such discontinuities, $u, p+B^2/\rho\mu_0$ and E should be continuous. In addition, if the transition is between régimes 1 and 2 no current sheet can exist and B must also be continuous. When the other conditions, both characteristic and algebraic, are added it is found that in all cases consistency is just satisfied.

Discontinuous transitions moving relative to the gas are studied by the method used by Germain (1960), Kulikovsky & Lyubimov (1959) and others. First of all small dissipative terms are introduced into the equations. These will have the effect of smoothing the transition so that the variables change steeply and continuously. The transition is now brought to rest locally by a suitable change of variables. It is assumed that the equations in this frame are dominated by the derivatives across the transition and the dissipative terms, so that steady-flow equations including these terms are adequate.

Dissipation is introduced by including viscosity terms from the Navier-Stokes equations and the effects of finite conductivity. The resulting equations are

$$\frac{d}{dx}(\rho u) = 0, \tag{4.1}$$

$$\frac{d}{dx}\left\{\rho u^2 + p + \frac{B^2}{2\mu_0}\right\} = \frac{4}{3}\frac{d}{dx}(\rho\nu u_x), \tag{4.2}$$

$$\frac{d}{dx}\left\{u\left(\tfrac{1}{2}\rho u^2 + \frac{\gamma p}{\gamma-1} - \tfrac{4}{3}\rho\nu u_x\right) + \frac{EB}{\mu_0}\right\} = 0, \tag{4.3}$$

$$E_x = 0, \tag{4.4}$$

$$-B_x/\mu_0 = J = \sigma(E-uB). \tag{4.5}$$

Equations (4.1), (4.2), (4.3) and (4.4) are integrated once to give

$$\rho u = C_1, \tag{4.6}$$

$$\rho u^2 + p + B^2/2\mu_0 = \tfrac{4}{3}\rho\nu u_x + C_2, \tag{4.7}$$

$$u\left(\tfrac{1}{2}\rho u^2 + \frac{\gamma p}{\gamma-1} - \tfrac{4}{3}\rho\nu u_x\right) + \frac{EB}{\mu_0} = C_3, \tag{4.8}$$

$$E = \text{const.}, \tag{4.9}$$

where C_1, C_2 and C_3 are constants. p and ρ are eliminated from equations (4.6), (4.7) and (4.8). Which leads to

$$u_x = \frac{3(\gamma-1)}{4\nu\gamma}\left[\frac{\gamma+1}{2}u^2 + (\gamma-1)\frac{C_3}{C_1} - (\gamma-1)\frac{EB}{\mu_0 C_1} - \frac{\gamma C_2 u}{C_1} + \frac{\gamma B^2 u}{2\mu_0 C_1}\right]. \tag{4.10}$$

From (4.5)

$$B_x = -\mu_0\sigma(E-uB). \tag{4.11}$$

Equations (4.10) and (4.11) are ordinary differential equations for u and B (E is a constant).

The only transitions that are of interest are those which link two uniform states, corresponding to solutions of (4.10) and (4.11) which start and end at points where B_x and u_x are zero. The points in the (B, u)-plane at which $u_x = 0$ lie on the curve

$$\frac{\gamma+1}{2}u^2 + (\gamma-1)\frac{C_3}{C_1} - (\gamma-1)\frac{EB}{\mu_0 C_1} - \frac{\gamma C_2 u}{C_1} + \frac{\gamma B^2 u}{2\mu_0 C_1} = 0. \tag{4.12}$$

B_x is zero either on the hyperbola

$$E - uB = 0, \tag{4.13}$$

or where $\sigma = 0$, that is $T < T^*$. The curve $T = T^*$ corresponds to $p/\rho = C^2$. The equation of this curve in the (B, u)-plane is obtained by using equations (4.6), (4.7) and (4.8) to express p and ρ in terms of u and B. It is

$$\tfrac{1}{2}u^2 C_1 - \frac{C_1 C^2}{\gamma-1} + \frac{B^2 u}{2\mu_0} - \frac{EB}{\mu_0} - uC_2 + C_3 = 0. \tag{4.14}$$

The curves $u_x = 0$, $B_x = 0$ divide the (B, u)-plane into a number of regions in which the signs of u_x and B_x are known. The curve $T = T^*$ further subdivides these regions with $B_x = 0$ for $T < T^*$, and $B_x \neq 0$ for $T > T^*$. The relative positions and shapes of the curves depend on the constants C_1, C_2, C_3 and E. A particular case is illustrated in figure 4. Only the positive quadrant of the (B, u)-plane is considered, since there are no relevant solution curves on which B or u change signs and cases where either u and or B are negative throughout are essentially the same as those with positive u and B. The point A is a saddle point and the nature of the solution curves in its neighbourhood are illustrated on the figure.

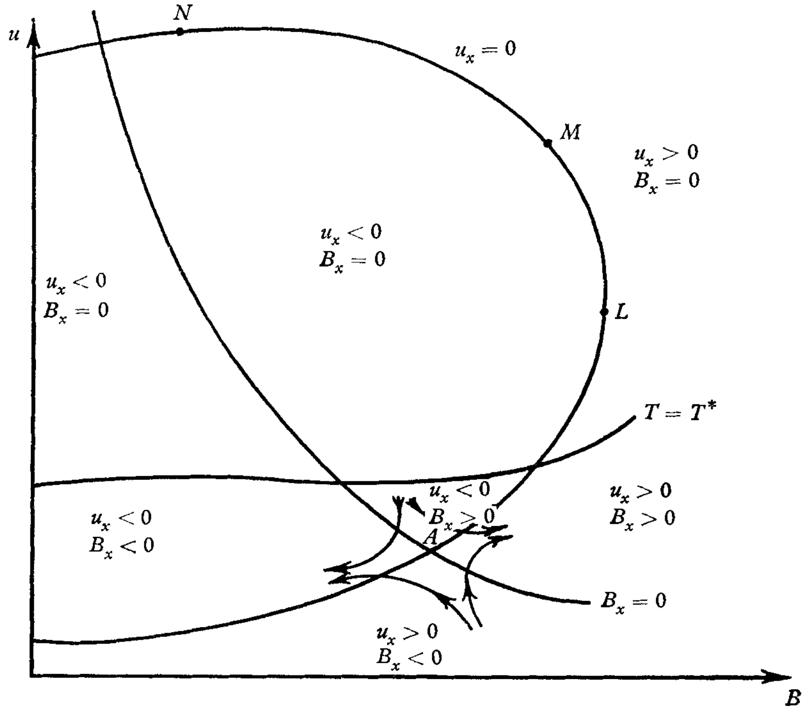

FIGURE 4. The behaviour of derivatives of u and B deduced from the structure equations.

The points L, M and N shown in figure 4 are the points on the curve $u_x = 0$ where u is equal to the wave speed in one of the three régimes. L is the point at which $u = a$. At this point $dB = 0$ along $u_x = 0$. M is the point where $u = c$. Geometrically it is the point where one of the family of curves $uB =$ const., touches $u_x = 0$. N is the point at which $u = -\lambda_1$ and is also the point where a member of the family of curves $T =$ const. touches $u_x = 0$.

There seem to be four distinct types of solution curve satisfying the required conditions. Three of these are illustrated in figure 5. The first is represented by the curve BA. This is the solution discussed by Kulikovsky & Lyubimov (1959) and by Soubbaramayer (1962). The gas is initially un-ionized and at first the magnetic field does not change while part of an ordinary gasdynamic shock transition takes place. At F the gas reaches the temperature T^* and starts to ionize. Over the remaining part of the solution curve both the velocity and the magnetic field change. At both of the end points B and A, $u_x = B_x = 0$. For any set of values of C_1, C_2, C_3 and E there is a unique starting-point B which leads to a solution reaching A. Ahead of the transition $u > a$ and behind it $u < c$.

The second type of solution is represented by the curve DE. At D the gas is un-ionized. It passes through part of a gasdynamic shock until it reaches $T = T^*$ at G. Subsequently current flows and the magnetic field increases. This increases the magnetic pressure and eventually causes the thermal pressure

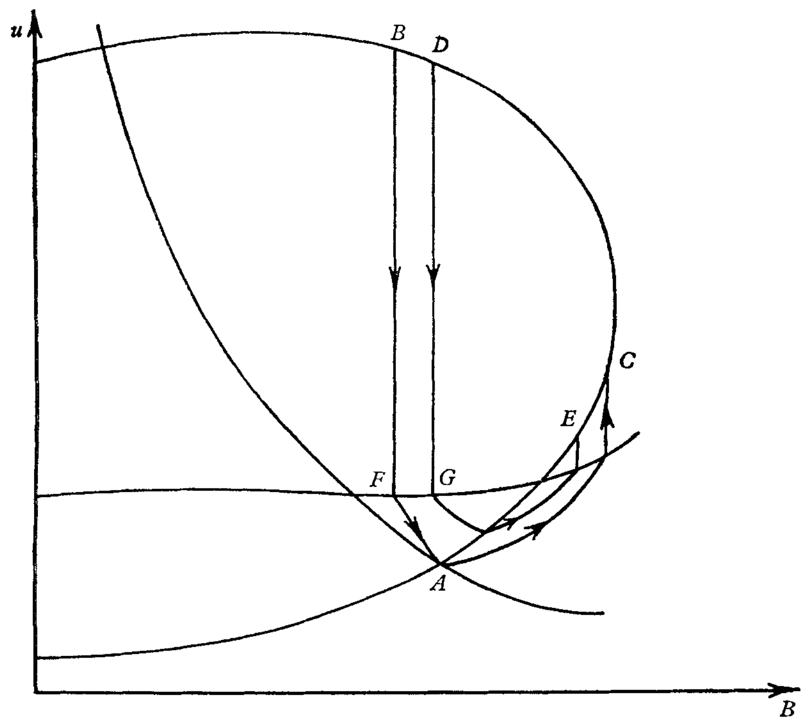

FIGURE 5. Three solution curves for the structure equations corresponding to transitions of types 1, 2 and 3.

and temperature to reduce so that the gas de-ionizes as the solution curve recrosses $T = T^*$. The velocity then increases with the magnetic field stationary until the condition $u_x = 0$ is reached at E. This transition is an arc moving through a magnetic field with un-ionized gas on either side. There is a relation between the positions of D and E on the $u_x = 0$ curve. At D, $u > a$ and at E, $u < a$.

The third type is represented by the curve AC. Initially the gas is fully ionized (régime 3). The solution leaves the saddle point A along a unique path with velocity and magnetic field increasing and temperature decreasing. When $T = T^*$ is reached the gas de-ionizes and the velocity increases until the conditions at the point C on the $u_x = 0$ curve are attained. For any set of values of C_1, C_2, C_3 and E there is a unique position for the point C. At A, $u < a$ and at C, $u < a$.

The fourth type of transition is shown in figure 6. (The axes are omitted for convenience.) The gas starts at H in régime 3 and the end state is at K where

$T = T^*$ and $u_x = 0$. The point H is a nodal point of the solution curves and K is a saddle point. Therefore the boundary conditions, that H is on $u_x = 0$ and that K is on $u_x = 0$ and $T = T^*$, are just sufficient to give a unique solution. Through this transition the gas passes from régime 3 to either régime 1 or 2. Ahead $u > c$, and behind $u < -\lambda_1$.

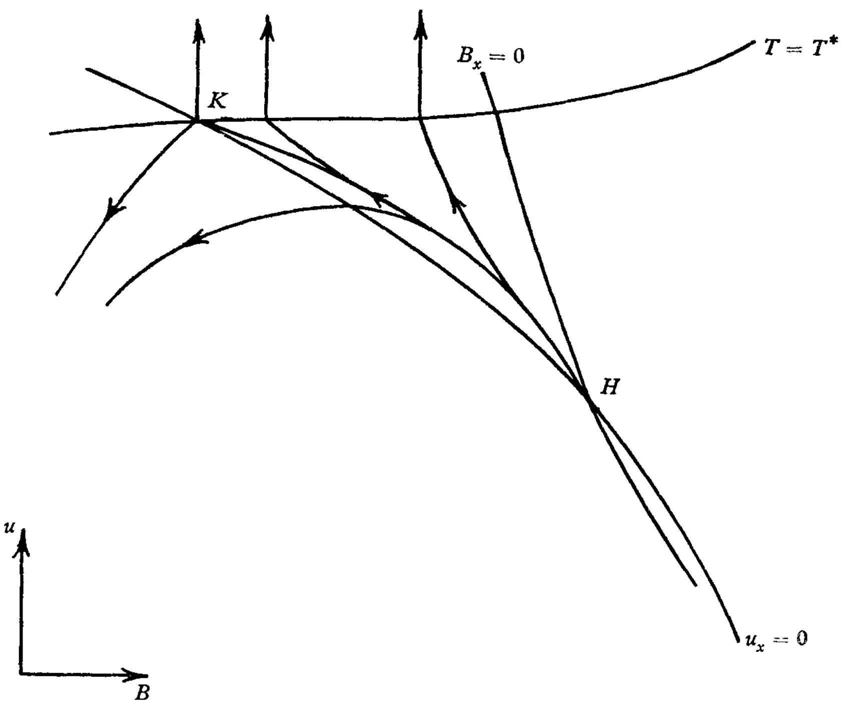

FIGURE 6. A solution curve corresponding to a transition of type 4.

Equations (4.6), (4.7), (4.8), (4.9) must be satisfied by both the end states $(\rho_1, u_1, p_1, B_1, E_1)$ and $(\rho_2, u_2, p_2, B_2, E_2)$ with $u_x = B_x = 0$. This gives four conservation relations between the end states:

$$\rho_1 u_1 = \rho_2 u_2,$$

$$\rho_1 u_1^2 + p_1 + \frac{B_1^2}{2\mu_0} = \rho_2 u_2^2 + p_2 + \frac{B_2^2}{2\mu_0},$$

$$u_1\left(\tfrac{1}{2}\rho_1 u_1^2 + \frac{\gamma p_1}{\gamma - 1}\right) + \frac{E_1 B_1}{\mu_0} = u_2\left(\tfrac{1}{2}\rho_2 u_2^2 + \frac{\gamma p_2}{\gamma - 1}\right) + \frac{E_2 B_2}{\mu_0},$$

$$E_1 = E_2.$$

In addition to these conditions the end states must satisfy any algebraic conditions relevant to the flow régimes to which they belong and internal structure conditions if the transition is of types 1, 2 or 3.

The procedure described in §3 can be applied to these four types of discontinuity. The results are summarized in table 2. It is found that types 1, 2 and 3 transitions can exist whenever the structure conditions are satisfied. Type 4 can exist in two forms, either as a $3 \to 1$ transition or as a $3 \to 2$ transition. In the second case, conditions behind it must be precisely sonic ($u = -\lambda_1$ in the steady frame).

Type	Régimes	Conditions ahead	Conditions behind
1	$1 \to 3$	$U > u+a$	$U < u+c$
2	$1 \to 1$	$U > u+a$	$U < u+a$
3	$3 \to 1$	$U < u+c$	$U < u+a$
4	Either $3 \to 1$	$U > u+c$	$U > u+a$
	or $3 \to 2$	$U > u+c$	$U = u+\lambda_2$

TABLE 2. Classification of shock-type transitions.

5. Boundary-value problems

The information obtained in §§3 and 4 can be used, in conjunction with knowledge of the behaviour within the individual régimes, to predict flow patterns for particular boundary-value problems. Since there are eight different types of continuous transitions and four types of discontinuities, in addition to the various continuous waves and shock waves which are possible in the individual régimes, many types of flow pattern are possible.

As an example, consider the following problem. A hot ionized gas ($T > T^*$) is confined in a rectangular tube between the poles of a magnet. It is assumed that the cross-section of the tube is much greater in a direction perpendicular to the magnetic field than it is in a direction parallel to the field, so that one-dimensional conditions may be assumed. It is also assumed that the magnetic field in the gas is initially uniform. One end of the tube is closed by a perfectly conducting piston which is initially at rest and is instantaneously accelerated at $t = 0$ and subsequently moves away from the gas at a constant velocity Q.

Various flow patterns may result depending on the values of the dimensionless parameters Q/C, $B_0^2/\rho_0\mu_0 a_0^2$, and Q/a_0. If Q is sufficiently small a simple régime 3 expansion wave moves into the gas, followed by a uniform flow region. This is illustrated in figure 7. For larger values of Q this flow pattern is not possible since the temperature of the gas decreases through the expansion and for sufficiently large values of Q would drop below $T = T^*$. For these values of Q a second type of flow results. This is illustrated in figure 8. The gas first passes through a régime 3 expansion, it then passes through a discontinuous transition to régime 1 (type 4 in table 2). This transition is followed by a uniform flow region, then a régime 1 expansion and finally a second uniform region. The temperature ahead of the transition is greater than T^* and it moves at precisely the wave speed ahead of it, so that it is also a characteristic of the régime 3 region. As Q is increased further, the discontinuity strengthens. The effect of the piston velocity is not communicated to the transition by sound waves in the region behind it, since these waves are too slow. There is a direct electromagnetic coupling between the conducting piston and the gas ahead of the transition. If Q is increased further, the condition that no magnetic field lines can pass through the perfectly conducting piston

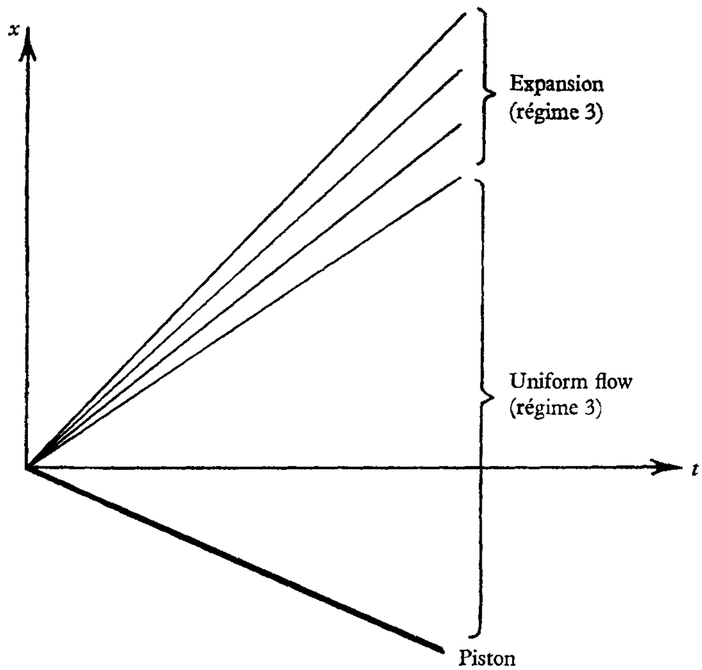

FIGURE 7. The flow pattern resulting when hot gas is expanded by moving a conducting piston at a low velocity.

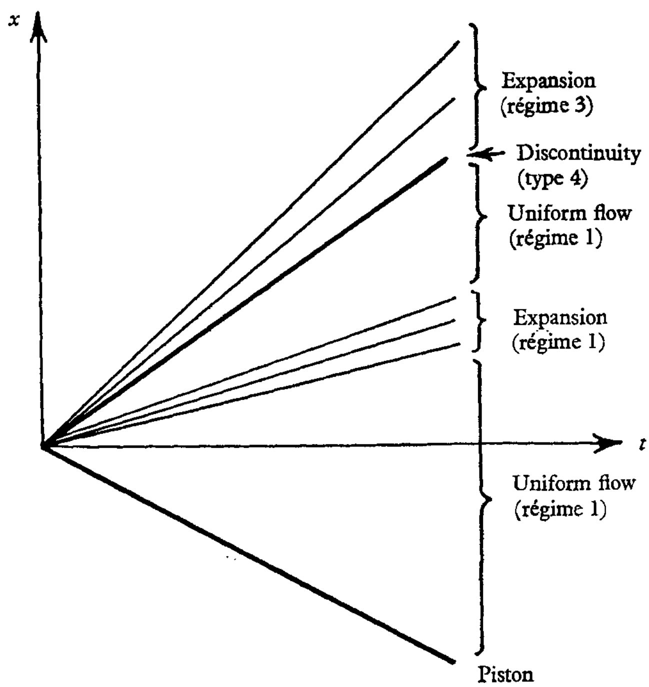

FIGURE 8. The flow pattern resulting when the gas is expanded by a moderately fast conducting piston.

requires lower magnetic fields between the piston and the transition. This in turn requires a stronger transition. Eventually a stage is reached when the $T = T^*$ curve touches the $u_x = 0$ curve in the (B, u)-plane diagram describing its structure, i.e. the points H and K of figures 4 and 6 coincide. No further increase in strength can occur. However, this is also the stage at which the type 4 transition can be regarded as a régime 3 to régime 2 transition, so for larger Q a régime 2

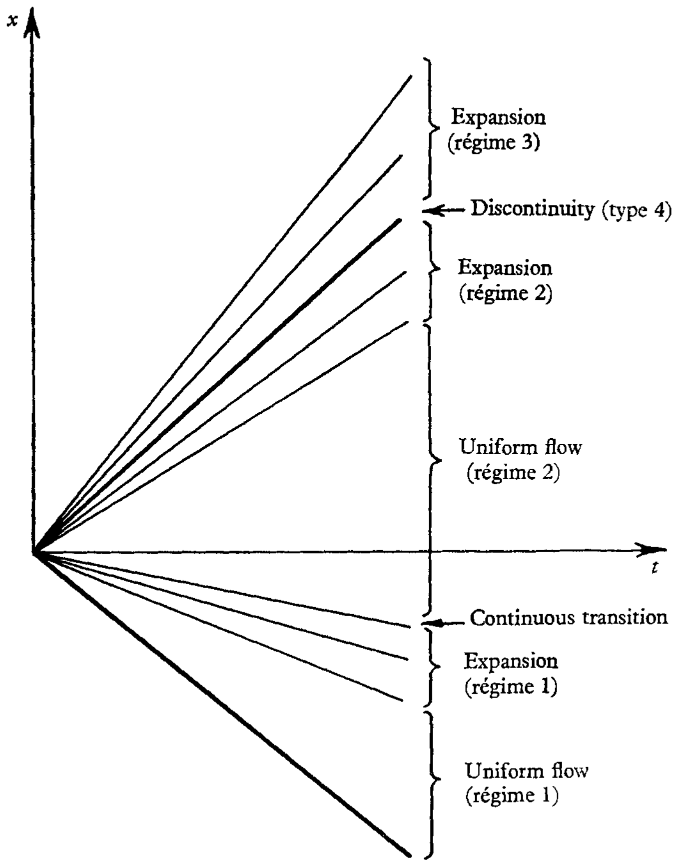

FIGURE 9. The fully-developed flow pattern resulting when the piston is fast moving.

region can occur behind the transition. This situation is illustrated in figure 9. The flow passes through a régime 3 expansion, a $3 \rightarrow 2$ discontinuity, a régime 2 expansion, a uniform régime 2 region, a continuous $2 \rightarrow 1$ transition followed immediately by a régime 1 expansion and finally a uniform régime 1 region.

There are thus three basically different flow patterns resulting from this simple boundary-value problem. In the case illustrated in figures 8 and 9, and for certain values of the parameters, the régime 1 expansion may expand the gas completely so that the final region becomes a vacuum. It is also possible that the discontinuous transition may move completely through the régime 3 expansion. In all, therefore, there are nine distinct patterns.

The case where the piston accelerates slowly can also be considered. A typical flow pattern for this case is shown in figure 10. Initially a simple régime 3 expansion wave moves into the gas. When the temperature on the piston decreases to T^* at the point A a discontinuous transition forms. Initially this has zero strength and is moving at the wave velocity $u+c$. As the piston accelerates further, it interacts magnetically with the gas in this front and drives the front

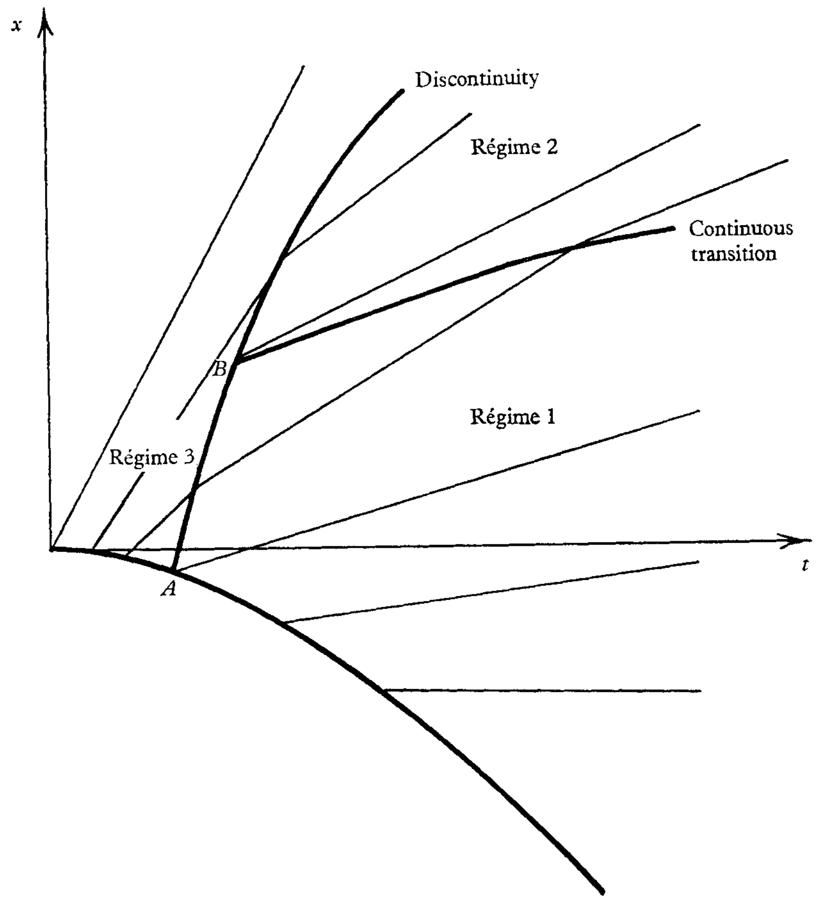

FIGURE 10. The flow pattern resulting when a conducting piston is accelerated continuously.

into the expansion. A régime 1 expansion moves into the gas from the piston and from the rear of the front. At the point B the front has accelerated so much that its velocity is sonic relative to the régime 2 wave speed behind it. At this point a continuous front originates, moving more slowly so that it is overtaken by the régime 1 expansion. The original front continues as a régime $3 \to 2$ front and a region of régime 2 flow exists between them. If the piston subsequently reaches a steady velocity the $3 \to 2$ front will be asymptotic to one of the régime 3 characteristics and the $2 \to 1$ front will be asymptotic to a régime 1 characteristic, and as $t \to \infty$ the flow becomes that depicted in figure 9.

The flow with a non-conducting piston can also be considered. In this case there is only one boundary condition on the piston, $u = -Q$. A unique solution can only be obtained if another condition involving B and or E is given for $x < 0$. A suitable condition can be obtained if it is assumed that the tube is 'strapped' at effectively $x = -\infty$ so that the magnetic field is constant on the side of the piston further from the gas. A typical flow pattern that can result if the piston is steadily accelerated is shown in figure 11.

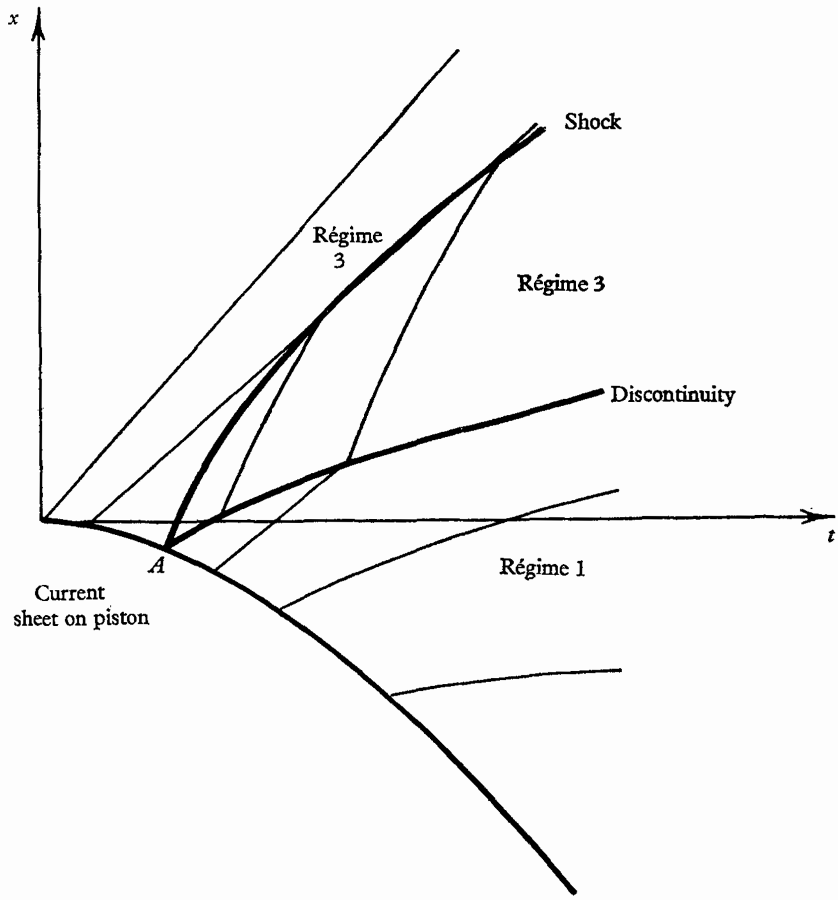

FIGURE 11. The flow pattern for a continuously accelerating non-conducting piston.

Initially a régime 3 expansion moves into the gas. This not only accelerates the gas to the piston velocity but reduces the magnetic field below the ambient value. A current sheet forms on the surface of the piston in order to match the field to ambient conditions behind the piston. When the point A is reached the expansion is strong enough to de-ionize the gas. At this point a régime $3 \to 3$ shock (see Marshall 1955) and a type 3, $3 \to 2$ transition start. Both these fronts initially have finite strengths which are such that their effects on the velocity cancel (the $3 \to 3$ shock compresses the gas which is then re-expanded by the $3 \to 1$ transition) but their combined effect on the magnetic field replaces that of the current sheet which exists up to A. The shock is supersonic relative to the régime 3 expansion ahead and therefore overtakes part of the expansion and is weakened. The $3 \to 1$ front is subsonic relative to the régime 1 flow behind it. As the piston accelerates further the front is overtaken by expansion waves which increase its strength. The $3 \to 1$ front is also subsonic relative to the régime 3 flow ahead of it, and can therefore send waves ahead which drive the $3 \to 3$ shock. If the piston

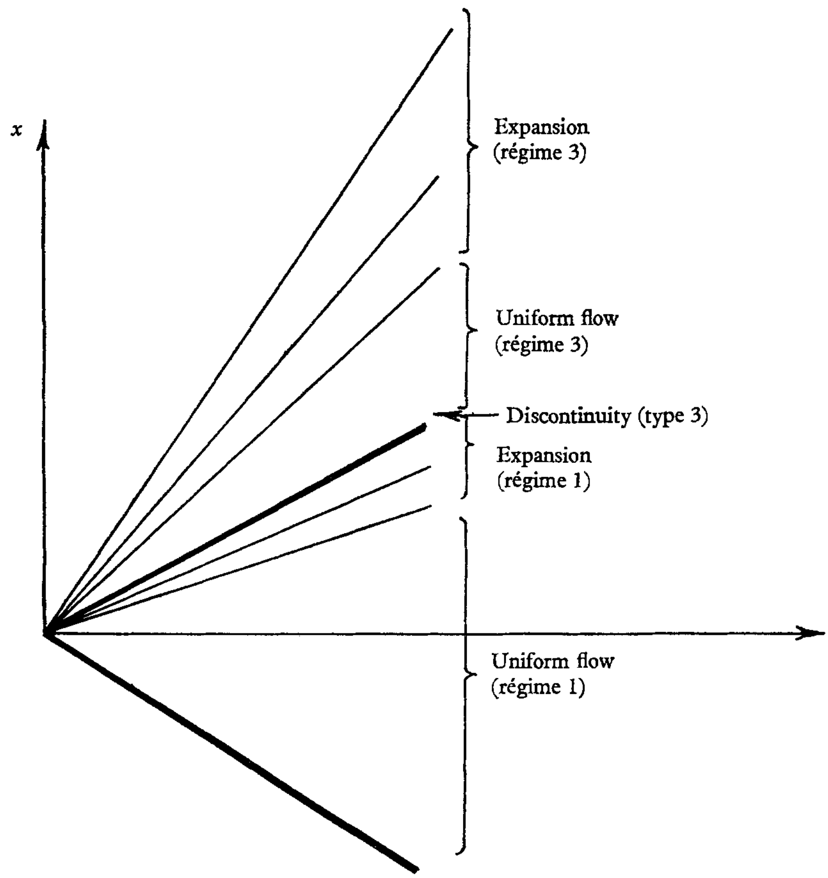

FIGURE 12. The fully-developed flow pattern for a non-conducting piston.

ultimately attains a uniform velocity the $3 \to 3$ shock becomes indefinitely weakened. The final flow pattern is that depicted in figure 12. This represents the fully-developed flow pattern. Other flow patterns arise if the piston velocity is not sufficiently large.

The two examples described above illustrate an important difference between the two possible types of discontinuous transition from régime 3 to régime 1 (table 2). In the conducting-piston case the front is driven electromagnetically and a type 4 front results and in the non-conducting piston case the front is driven by expansion waves in the gas and a type 3 front is formed.

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